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ALGEBRA.

76. Proposed by E. B. ESCOTT, Fellow in Mathematics, University of Chicago, Chicago, Ill.

Prove the identities

$$2 - \sqrt{2} = \frac{1}{2} + \frac{1}{2^2 \cdot 3} + \frac{1}{2^3 \cdot 3 \cdot 17} + \frac{1}{2^4 \cdot 3 \cdot 17 \cdot 577} \dots$$

$$\frac{5-\sqrt{5}}{2} = 1 + \frac{1}{3 + \frac{1}{3.7 + \frac{1}{3.7.47 + \frac{1}{3.7.47.2207 + \dots}}}}$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let N be any number, then

$$N = \frac{Np^m}{p^m} \cdot \frac{r^m}{r^m} = \frac{p^m}{r^m} \left[1 - \left(\frac{p^m - Nr^m}{p^m} \right) \right].$$

$$\therefore \sqrt[m]{N} = \frac{p}{r} \left[1 - \frac{p^m - Nr^m}{p^m} \right]^{1/m} = \frac{p}{r} \left[1 - \left(\frac{p^m - Nr^m}{mp^m} \right) \right]$$

$$- \frac{(m-1)}{1.2} \left(\frac{p^m - Nr^m}{mp^m} \right)^2 - \frac{(m-1)(2m-1)}{1.2.3} \left(\frac{p^m - Nr^m}{mp^m} \right)^3$$

$$- \frac{(m-1)(2m-1)(3m-1)}{1.2.3.4} \left(\frac{p^m - Nr^m}{mp^m} \right)^4 - \text{etc.}].$$

Let $m=2$ and $p^m-Nr^m=1$,

$$\therefore \sqrt{N} = \frac{p}{r} \left(1 - \frac{1}{p^2} \right)^{\frac{1}{2}} = \frac{p}{r} \left[1 - \frac{1}{2} \left(\frac{1}{p^2} \right) - \frac{1}{2^3} \left(\frac{1}{p^2} \right)^2 - \frac{1}{2^4} \left(\frac{1}{p^2} \right)^3 - \frac{5}{2^7} \left(\frac{1}{p^2} \right)^4 - \text{etc.} \right] \dots \quad (1)$$

Let $m=2$ and $p^m-Nr^m=4$,

$$\therefore \sqrt{N} = \frac{p}{r} \left(1 - \frac{4}{p^2} \right)^{\frac{1}{2}} = \frac{p}{r} \left[1 - \frac{1}{2} \left(\frac{4}{p^2} \right) - \frac{1}{1 \cdot 2 \cdot 2^2} \left(\frac{4}{p^2} \right)^2 - \frac{1.3}{1 \cdot 2 \cdot 3 \cdot 2^3} \left(\frac{4}{p^2} \right)^3 - \frac{1.3 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2^4} \left(\frac{4}{p^2} \right)^4 - \text{etc.} \right].$$

$$\therefore \sqrt{N} = \frac{p}{r} \left[1 - \frac{2}{p^2} - \frac{2}{p^4} - \frac{4}{p^6} - \frac{2.5}{p^8} - \frac{4.7}{p^{10}} - \frac{3.4.7}{p^{12}} - \text{etc.} \right] \dots (2).$$

In (1) let $N=2$, $p=17$, $r=12$, $\therefore p^2 - 2r^2 = 1$.

$$\therefore \sqrt{2} = 1\frac{1}{3}(1 - \frac{1}{2 \cdot 17^2} - \frac{1}{2^3 \cdot 17^4} - \frac{1}{2^4 \cdot 17^6} - \frac{5}{2^7 \cdot 17^8} - \text{etc.}).$$

$$\therefore 2 - \sqrt{2} = 2 - \frac{1}{2} + \frac{1}{2 \cdot 12 \cdot 17} + \frac{1}{2^3 \cdot 12 \cdot 17^3} + \frac{1}{2^4 \cdot 12 \cdot 17^5} + \frac{5}{2^7 \cdot 12 \cdot 17^7} + \text{etc.}$$

$$2 - \sqrt{2} = \frac{1}{2} + \frac{1}{2 \cdot 12 \cdot 17} + \frac{1}{2^3 \cdot 12 \cdot 17^3} + \frac{1}{2^4 \cdot 12 \cdot 17^5} + \frac{5}{2^7 \cdot 12 \cdot 17^7} + \text{etc.}$$

$$= \frac{1}{2} + \frac{1}{2^2 \cdot 3} + \frac{1}{2^3 \cdot 3 \cdot 17} + \left(\frac{1}{2^4 \cdot 3 \cdot 17 \cdot 577} - \frac{1}{2^4 \cdot 3 \cdot 17 \cdot 577 \cdot 578} \right)$$

$$+ \left(\frac{1}{2^4 \cdot 3 \cdot 17 \cdot 577 \cdot 578} - \frac{1}{2^4 \cdot 3 \cdot 17 \cdot 577 \cdot 578^2} \right) + \text{etc.}$$

$$+ \left(\frac{1}{2^4 \cdot 3 \cdot 17 \cdot 577 \cdot 578^2} + \frac{1}{2^5 \cdot 3 \cdot 17 \cdot 577 \cdot 665857} - \right) + \text{etc.}$$

$$\therefore 2 - \sqrt{2} = \frac{1}{2} + \frac{1}{2^2 \cdot 3} + \frac{1}{2^3 \cdot 3 \cdot 17} + \frac{1}{2^4 \cdot 3 \cdot 17 \cdot 577} + \frac{1}{2^5 \cdot 3 \cdot 17 \cdot 577 \cdot 665857} + \dots$$

In (2) let $N=5$, $p=7$, $r=3$, $\therefore p^2 - 5r^2 = 4$.

$$\therefore \sqrt{5} = \frac{1}{3} \left(1 - \frac{2}{7^2} - \frac{2}{7^4} - \frac{4}{7^6} - \frac{2.5}{7^8} - \frac{4.7}{7^{10}} - \text{etc.} \right)$$

$$\sqrt{2} = \frac{5}{7} - \frac{1}{3.7} - \frac{1}{3.7^3} - \frac{2}{3.7^5} - \frac{5}{3.7^7} - \frac{2}{3.7^8} - \text{etc.}$$

$$\frac{5 - \sqrt{5}}{2} = \frac{5}{2} - \frac{1}{6} + \frac{1}{3.7} + \frac{1}{3.7^3} + \frac{2}{3.7^5} + \frac{5}{3.7^7} + \frac{2}{3.7^8} + \text{etc.} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3.7}$$

$$+ \left(\frac{1}{3.7.47} - \frac{2}{3.7.47.49} \right) + \left(\frac{2}{3.7.47.49} - \frac{4}{3.7.47.49^2} \right) + \left(\frac{4}{3.7.47.49^2} + \frac{1}{3.7.47.2207} - \right) + \text{etc.}$$

$$\therefore \frac{5 - \sqrt{5}}{2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3.7} + \frac{1}{3.7.47} + \frac{1}{3.7.47.2207} \dots$$

The above formulæ were used by Dr. Artemas Martin for extracting the root of numbers several years ago.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Since $2 - \sqrt{2}$ is rather more than $\frac{1}{2}$, we can, confining ourselves to 4 terms, put $2 - \sqrt{2} = \frac{1}{2} + \frac{1}{2^2 x} + \frac{1}{2^3 xy} + \frac{1}{2^4 xyz}$, or by transposition, $\sqrt{2} = \frac{3}{2} - \frac{1}{4x} - \frac{1}{8xy} - \frac{1}{16xyz}$. Squaring and omitting the term $\frac{1}{256x^2y^2z^2}$, we get, after multiplying by 64 x ,

$$16x = 48 - \frac{4}{x} + \frac{24}{y} - \frac{1}{xy^2} + \frac{12}{yz} - \frac{4}{xy} - \frac{2}{xyz} - \frac{1}{xy^2z},$$

whence $x=3$, and substituting this value and multiplying by $3y$, we get

$$4y = 68 - \frac{1}{y} + \frac{34}{z} - \frac{1}{yz};$$

$\therefore y=17$, and substituting it, and multiplying by $17z$, we get $z=578-1=577$. If another term had been desired we would have annexed to the above series the term $\frac{1}{2^5xyzt}$ and proceeded in the same manner, though for 4 terms, the series is sufficiently convergent to render the value of $2-\sqrt{2}$ correct for at least 8 decimals.

Since $\frac{5-\sqrt{5}}{2}$ is $=1+$ fraction, we put, restricting the series to 4 terms besides 1,

$$\frac{5-\sqrt{5}}{2} = 1 + \frac{1}{x} + \frac{1}{xy} + \frac{1}{xyz} + \frac{1}{xyzt},$$

$$\text{whence } \sqrt{5} = 3 - \frac{2}{x} - \frac{2}{xy} - \frac{2}{xyz} - \frac{2}{xyzt}.$$

Squaring and omitting the term $4/x^2y^2z^2t^2$, we have, transposing, suppressing the common factor 4, and multiplying by x ,

$$x = 3 - 1/x + 3/y - 1/xy^2 - 1/xy^2z^2 + 3/yz + 3/yzt$$

$$- 2/xy - 2/xyz - 2/xyzt - 2/xy^2z - 2/xy^2z^2t, \text{ whence } x=3.$$

Substituting this value, and multiplying by $3y$, we have

$$y = 7 - 1/y - 1/yz^2 + 7/z + 7/zt - 2/yz - 2/yz^2t; \text{ whence } y=7.$$

Substituting, and multiplying by $7z$, we get

$$z = 47 - 1/z + 47/t - 2/zt; \text{ whence } z=47.$$

Substituting, and multiplying by $47t$, we get $t=2207$.

Such series converge very rapidly. In German works they are called "Theilbruchreihen," signifying *partial fraction series*. Every common fraction may be converted into such a series by the following process.

Let the fraction be $\frac{15}{8}$.

15)19(2

$$\therefore \frac{15}{8} = \frac{1}{2} + 1/2.2 + 1/2.2.7 + 1/2.2.7.10 + 1/2.2.7.10.19,$$

11)19(2

so that, taking successively 1, 2, 3, 4 of these, we get
as in continuous fractions, approximate values, as $\frac{1}{2}$,
 $\frac{3}{4}$, $\frac{11}{14}$, $\frac{22}{28}$. As in continuous fractions, the numerator
of the difference of any two consecutive approximate fractions is always =1. Supposing the indeterminate equation $15x-19y=1$. Changing $\frac{15}{8}$ by the above method into such a

3)19(7

2)19(10

1)19(19

series, and take the last approximate fraction, viz, $\frac{22}{280}$, $x=280$, $y=221$, will furnish two values; also, the preceding fraction $\frac{11}{14}$, $x=14$, $y=11$.

77. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, N. H.

Solve the equation, $(6x^2 + x - 3)^2 - 48^2 = (x + 15)^2$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and A. H. BELL, Hillsboro, Ill.

$$(6x^2 + x - 3)^2 - (48)^2 = (x + 15)^2.$$

$$\therefore 3x^4 + x^3 - 3x^2 - 3x - 210 = 0.$$

$$\therefore x_1 = 2.924412 + ,$$

$$x_2 = -3.041623 - ,$$

$$x_3 = 2.804395\sqrt{-1} - .158044.$$

$$x_4 = -2.804395\sqrt{-1} - .158044.$$

II. Solution by J. MARCUS BOORMAN, Consultative Mechanician, Counsellor at Law, Inventor, Etc., Hewlett, Long Island, New York.

The real roots of $x^4 + \frac{1}{3}x^3 - x^2 - x = 70$ (the forms to which the given equation reduces) are :

$$\begin{aligned} x &= +2.924412 \quad 149966 \quad 623189 \quad 6108(58211) \\ x &= -3.041622 \quad 694570 \quad 750484 \quad 61819(75892) \end{aligned} \quad \left. \begin{array}{l} \text{true to " marks, -probably} \\ \text{25 decimals true.} \end{array} \right\}$$

Found thus: $x^4 + \frac{1}{3}x^3 - 1x^2 - 1x - 70$
At sight $x = \pm 3$ (near), try -3.04

$$\begin{array}{r} \underline{-2.71 - x^3} \\ +8.24 - \end{array} \quad \begin{array}{r} \underline{-22.01} \\ 69.9504 \end{array}$$

$$\begin{array}{r} \underline{+7.24 -} \\ -23.01 \end{array} \quad \begin{array}{r} \underline{0.04 -} \\ \end{array}$$

$$\begin{array}{r} \text{Multiply by } -3, -.04 \quad +.13 \\ .1084 + \end{array} \quad \begin{array}{r} 21.72 \quad 69.03 \\ 29 - \quad .9204 \end{array} \quad \begin{array}{r} \text{error + *} \\ \underline{-22.01} \quad 69.9504 \end{array}$$

$$\begin{array}{r} \underline{+8.24 -} \\ \end{array}$$

*The root \therefore is $-2.0316 +$ nearly.

Like treatment by $+2.9$ gives 2.925 near true. I get the rest in a more concise way (shorter than Horner's).

The resulting equations of above roots are : (the coefficients appear on face of computation)

$$0 = x^3 + (+\text{root} + \frac{1}{3})x^2 + 8.526990 \quad 472861 \quad 28178 \text{etc. } x + 23.936434, 541435, 17396 + .$$

$$0 = x^3 + (-\text{root} + \frac{1}{3})x^2 + 7.237594 \quad 384604 \quad 24939 \text{etc. } x - 23.914031, 334310, 10968 + .$$

$$0 = (5.966034, 84 \text{etc.})x^2 + (1.289496, 088257, 03238 +)x + 46.950465, 875795, 28364 + .$$

Hence $x^2 + 0.216122, 788722x = -7.869626, 49385$ etc.

$\therefore x_1 = -0.108$ etc. $\pm \sqrt{-7.890 + .}$

Shorter and more accurate than Horner's method, especially for 5th degree and 6th degree equations.

The other two roots are imaginary.